

# ITERATED QUASI-ARITHMETIC MEAN-TYPE MAPPINGS

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**ABSTRACT.** For a family of quasi-arithmetic means satisfying certain smoothness condition we majorize the speed of convergence of the iterative sequence of self-mappings having a mean on each entry, described in the definition of Gaussian product, to relevant mean-type mapping. We apply this result to approximate any continuous function which is invariant with respect to such a self-mappings.

## 1. INTRODUCTION

Iterative selfmappings frequently appears in the theory of fixed point and dynamical systems. In the present paper we will deal with selfmappings build up by quasi-arithmetic means.

The idea of quasi-arithmetic means was formally introduced in a series of nearly simultaneous papers [7, 11, 1] as a natural generalization of power means. These means have been extensively dealt with ever since its introduction in the early 1930s; cf. e.g. [3, chap. 4]. Many results concerning power means have its corresponding facts concerning this family (frequently under some additional assumption).

In this spirit we turn into Gauss' concept of arithmetic-geometric mean [5]. This idea was generalized many times. Let me mention the results of Gustin [6], generalizing this process to the family of power means with some additional weights, and by Matkowski [8], who proved that this compound could be introduced for a vast family of means (in particular - all quasi-arithmetic means). In the present paper we are going to adopt this idea to a family of quasi-arithmetic means satisfying some smoothness conditions.

Our main result, worded exactly in Theorem 2 (section 2.2), assert that the difference between the maximal and minimal entry of vector in each iteration can be effectively majorize. For a family of quasi-arithmetic means satisfying some smoothness conditions, this difference tends to zero quadratically (Lemma 4.3).

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In case of arithmetic-geometric mean such an estimation has already been given by Gauss in his famous [5] (see also (2.2) below). Our result (worded in Theorem 1 and in optimized version in Theorem 2) gives, regrettably, worse estimation than Gauss' one, however for much more general family of means.

The crucial tool in the present note will be the operator introduced by Mikusiński and, independently, Łojasiewicz in the first post-war issue of *Studia Mathematica* [10]. We require not only the weakest possible assumption to define this technically crucial tool - operator  $f''/f'$  - in our notion such an assumption is represented by set  $\mathcal{S}$ . We will also claim absolute boundedness of this operator (set  $\mathcal{S}_K$ ). This assumption could be omitted in some nonrestrictive way, what will be briefly described in section 2.3.

At the moment we are going to introduce necessary definitions and corresponding results (section 2.1) as well as present our main results (section 2.2). These results are then applied in section 3, while their proofs are postponed until section 5. Most of technical details were extracted from the proofs and are presented independently, in section 4.

## 2. MAIN RESULT

**2.1. Preliminaries and overview.** For any continuous, strictly monotone function  $f: I \rightarrow \mathbb{R}$  ( $I$  - an interval) and any vector  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k) \in I^k$ ,  $k \in \mathbb{N}$  we define

$$A_{[f]}(\mathbf{x}) := f^{-1} \left( \frac{f(\mathbf{x}_1) + f(\mathbf{x}_2) + \dots + f(\mathbf{x}_k)}{k} \right).$$

In our setting we will fix  $k \in \mathbb{N}$  and consider a family of continuous, strictly monotone functions  $\mathbf{f} = (f_1, f_2, \dots, f_k)$ ,  $f_j: I \rightarrow \mathbb{R}$ ,  $j \in \{1, \dots, k\}$ ,  $I$  - an interval. It will lead us to, at first, family of mappings  $A_{[f_j]}: I^k \rightarrow I$  and, later, a selfmapping  $\mathbf{A}_{[\mathbf{f}]}: I^k \rightarrow I^k$  being its product

$$\mathbf{A}_{[\mathbf{f}]}(\mathbf{x}) := \left( A_{[f_1]}(\mathbf{x}), \dots, A_{[f_k]}(\mathbf{x}) \right).$$

Matkowski proved [8] that, under some general conditions, there exists a unique function  $M_{[\mathbf{f}]}: I^k \rightarrow I$  satisfying (i)  $M_{[\mathbf{f}]} \circ \mathbf{A}_{[\mathbf{f}]} = M_{[\mathbf{f}]}$  and (ii)  $\min(\mathbf{x}) \leq M_{[\mathbf{f}]}(\mathbf{x}) \leq \max(\mathbf{x})$  for any  $\mathbf{x} \in I^k$ . He also proved that

$$M_{[\mathbf{f}]}(\mathbf{x}) = \liminf_{n \rightarrow \infty} \left[ \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) \right]_i = \limsup_{n \rightarrow \infty} \left[ \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) \right]_i, \quad \mathbf{x} \in I^k, i \in \{1, \dots, k\}.$$

It immediately implies

$$(2.1) \quad \lim_{n \rightarrow \infty} \left( \max \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) \right) = 0, \quad \mathbf{x} \in I^k.$$

By setting  $\mathbf{M}_{[\mathbf{f}]} = (M_{[\mathbf{f}]}, \dots, M_{[\mathbf{f}]})$  one gets  $\mathbf{A}_{[\mathbf{f}]}^n \xrightarrow[n \rightarrow \infty]{} \mathbf{M}_{[\mathbf{f}]}$  pointwise.

We are going to prove whenever  $f_j: I \rightarrow \mathbb{R}$ ,  $j \in \{1, \dots, k\}$  satisfies some smoothness conditions then the limit in (2.1) not only equals 0, but also the speed of convergence can be effectively majorize.

Such a result is known for the famous arithmetic-geometric iteration. Let us consider two positive numbers  $a, b > 0$ . Let  $a_0 = a$ ,  $b_0 = b$  and  $a_{n+1} = \frac{1}{2}(a_n + b_n)$ ,  $b_{n+1} = \sqrt{a_n b_n}$ . Gauss [5] proved that these sequences converge and have a common limit. This limit is used to called arithmetic-geometric mean ( $\mathcal{AGM}$ ) of  $a$  and  $b$ . It is known, [2, p.354], that

$$(2.2) \quad a_{n+1}^2 - b_{n+1}^2 < \left( \frac{a_n^2 - b_n^2}{4\mathcal{AGM}(a, b)} \right)^2.$$

So not only  $a_n - b_n \rightarrow 0$ , but we can prove that it converges quadratically. Our result, worded exactly in Theorem 2, asserts that this speed of convergence is natural for quasi-arithmetic means generated by functions satisfying some smoothness condition.

Now we turn into the result of Mikusiński [10]. He, and independently Łojasiewicz (compare [10, footnote 2]), expressed handy tool to compare means in terms of operator  $P_f := f''/f'$ . More precisely their result reads

**Proposition 2.1** (Basic comparison). *Let  $I$  be an interval,  $f, g \in \mathcal{C}^2(I)$ ,  $f' \cdot g' \neq 0$  on  $I$ . Then the following conditions are equivalent:*

- (i)  $A_{[f]}(\mathbf{x}) \geq A_{[g]}(\mathbf{x})$  for all vectors  $\mathbf{x} \in I^n$ ,  $n \in \mathbb{N}$  with both sides equal only when  $\mathbf{x}$  is a constant vector.
- (ii)  $P_f > P_g$  on a dense subset of  $I$ ,
- (iii)  $(\operatorname{sgn} f') \cdot (f \circ g^{-1})$  is strictly convex,

The operator  $P$  is so central in our consideration that we will assume that the considered function are smooth enough to use it. Moreover we will claim some additional assumption. More precisely let

$$\mathcal{S}(I) := \{f \in \mathcal{C}^2(I) : f' \neq 0 \text{ and } f'' \text{ has a locally bounded variation} \}.$$

Very often we will use global estimation of  $f''/f'$  and, as it is handy, for  $K > 0$ , we put

$$\mathcal{S}_K(I) := \{f \in \mathcal{S}(I) : \|f''/f'\|_\infty \leq K\}.$$

This assumption is deeply connected with family of log-exp means (cf. [3, p. 269]) defined for any  $a \in \mathbb{R}^k$ ,  $k \in \mathbb{N}_+$  as

$$\mathcal{E}_p(a) := \begin{cases} \frac{1}{p} \ln \left( \frac{e^{p \cdot a_1} + e^{p \cdot a_2} + \dots + e^{p \cdot a_k}}{k} \right) & p \neq 0, \\ \frac{1}{n} (a_1 + \dots + a_n) & p = 0. \end{cases}$$

Indeed, slightly weaker version of Proposition 2.1 ascertain that

$$(2.3) \quad \mathcal{S}_K(I) = \{f \in \mathcal{S}(I) : \mathcal{E}_{-K} \leq A_{[f]} \leq \mathcal{E}_K\}.$$

**Remark 1.** *Theorems below will be valid for functions belonging to  $\mathcal{S}_K$  for some  $K$ . It is important to note that the result depends only on input vector  $\mathbf{x}$  and a number  $K$ . In particular the number of functions, as well as functions themselves are not essential.*

**2.2. Formulation.** At the moment we are going to present a precise estimation of the speed of convergence. This theorem below will depend on a free parameter  $l$ . There is no universal (optimal) value of  $l$  that could be plugged into this theorem, the most natural possibility will be presented immediately after.

**Theorem 1.** *Let  $I$  be an interval;  $k \in \mathbb{N}$ ;  $K \in (0, +\infty)$  and  $\mathbf{f} = (f_1, f_2, \dots, f_k)$  be a family of functions,  $f_i \in \mathcal{S}_K(I)$  for any  $i \in \{1, \dots, k\}$ .*

*Let  $\alpha = \frac{3+7e}{3}$  [ $\alpha \approx 7.34$ ]. Then*

$$\max \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) < \frac{1}{\alpha K} (\alpha l)^{2^{n-n_0}}$$

*for any  $\mathbf{x} \in I^k$ ;  $l \in (0, 1)$  and*

$$n \geq \left\lceil \log_2 \left( \frac{\exp(K(\max \mathbf{x} - \min \mathbf{x})) - 1}{e^l - 1} \right) \right\rceil =: n_0.$$

In the result below we minimize the value on the right hand of the main inequality - it is a very natural challenge. However, if we would like to decrease  $n_0$ , we may change value of  $l$ .

Minimalization of the right hand side is realized for  $l \approx 0.05$  ( $l = \xi$  is the setting of theorem below). Therefore, we get the following

**Theorem 2.** *Let  $I$  be an interval;  $k \in \mathbb{N}$ ;  $K \in (0, +\infty)$  and  $\mathbf{f} = (f_1, f_2, \dots, f_k)$  be a family of functions,  $f_i \in \mathcal{S}_K(I)$  for any  $i \in \{1, \dots, k\}$ .*

*Let  $\alpha = \frac{3+7e}{3}$ ;  $\mu$  be a minimum value of a function  $(0, 1) \ni l \mapsto (\alpha l)^{(e^l-1)/2}$  achieving for  $l = \xi$  [ $\alpha \approx 7.34$ ;  $\mu \approx 0.97$ ;  $\xi \approx 0.05$ ]. Then*

$$\max \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) < \frac{1}{\alpha K} \mu^{\frac{2^n}{\exp(K(\max \mathbf{x} - \min \mathbf{x})) - 1}}$$

*for any  $\mathbf{x} \in I^k$  and*

$$n \geq \log_2(e) \cdot K \cdot (\max \mathbf{x} - \min \mathbf{x}) - \log_2(e^\xi - 1) + 1 =: n_1$$

$$[\text{approx. } n_1 \approx 1.443 \cdot K \cdot (\max \mathbf{x} - \min \mathbf{x}) + 5.25].$$

Relevant proofs of these theorems will be postponed until section 5, as in the proof we need some lemmas of section 4.

**2.3. Possible reformulation.** In both theorems we can restrict interval  $I$  to  $[\min \mathbf{x}, \max \mathbf{x}]$  and assume the functions belongs to  $\mathcal{S}(I)$  [taking  $K$  - the best possible].

More precisely, we can change the order of assumptions in the following way: First, take an interval  $I$ , a natural number  $k$ , and  $k$ -tuple  $\mathbf{f} = (f_1, f_2, \dots, f_k)$ ,  $f_i \in \mathcal{S}(I)$  for any  $i \in \{1, \dots, k\}$ . Then, for  $\mathbf{x} \in I^k$ , we *define*

$$K := \sup_{\substack{x \in [\min \mathbf{x}, \max \mathbf{x}] \\ i \in \{1, \dots, k\}}} |P_{f_i}(x)|.$$

Such a reformulation is natural but (i) we need to calculate  $K$ , which could be difficult and (ii) Remark 1 voids. However we will apply this procedure in section 3.2.

### 3. APPLICATIONS

In this section we are going to present two, fairly different, applications. First one, corresponding with earlier result of Matkowski, we are going to prove possible way to estimate a function, which are invariant under self-mapping  $\mathbf{A}_{[\mathbf{f}]}$ . Second one is an application of Theorem 2 in majorization of the difference between arithmetic-geometric mean and well-known iteration procedure.

#### 3.1. Diagonally continuous, invariant functions.

**Theorem 3.** *Let  $I$  be an interval;  $k \in \mathbb{N}$ ;  $K \in (0, +\infty)$  and let  $\mathbf{f} = (f_1, f_2, \dots, f_k)$ , where  $f_i \in \mathcal{S}_K(I)$  for any  $i \in \{1, \dots, k\}$ . Then*

*([9]) A function  $F: I^k \rightarrow I$  continuous on a diagonal  $\Delta := \{(x, \dots, x): x \in I\}$  satisfies the functional equation*

$$F(\mathbf{x}) = F(A_{[f_1]}(\mathbf{x}), \dots, A_{[f_k]}(\mathbf{x})), \quad \mathbf{x} \in I^k,$$

*iff*

$$F(\mathbf{x}) = \varphi \circ M_{[\mathbf{f}]}(\mathbf{x}), \quad \mathbf{x} \in I^k,$$

*where  $M_{[\mathbf{f}]}$  is the Gaussian product of the mean  $(A_{[f_1]}, A_{[f_2]}, \dots, A_{[f_k]})$  and  $\varphi$  is an arbitrary continuous function.*

*Moreover, if  $\alpha$ ,  $\mu$  and  $n_1$  are like in Theorem 2 and  $\varphi: I \rightarrow \mathbb{R}$  is a function of the modulus continuity  $\omega_\varphi$ , we have*

$$\left| F(\mathbf{x}) - \varphi \left( [\mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x})]_i \right) \right| \leq \omega_\varphi \left( \frac{1}{\alpha K} \mu^{\frac{2^n}{\exp(K(\max \mathbf{x} - \min \mathbf{x})) - 1}} \right)$$

*for any  $n > n_1$ ;  $i \in \{1, \dots, k\}$  and  $\mathbf{x} \in I^k$ . ,  $\omega_\varphi$  is a modulus of continuity.*

*Proof.* Fix any  $\mathbf{x} \in I^k$  and  $n \geq n_0$ . We know that

$$M_{[\mathbf{f}]}(\mathbf{x}) \in [\min \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}), \max \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x})].$$

Then, by Theorem 2, one has

$$\begin{aligned} \left| M_{[\mathbf{f}]}(\mathbf{x}) - [\mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x})]_i \right| &\leq \max \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) \\ &\leq \frac{1}{\alpha K} \mu^{\frac{2^n}{\exp(K(\max \mathbf{x} - \min \mathbf{x}) - 1)}}, i \in \{1, \dots, k\}. \end{aligned}$$

Whence, for any  $i \in \{1, \dots, k\}$

$$\begin{aligned} \left| F(\mathbf{x}) - \varphi \left( [\mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x})]_i \right) \right| &= \left| \varphi \circ M_{[\mathbf{f}]}(\mathbf{x}) - \varphi \left( [\mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x})]_i \right) \right| \\ &\leq \omega_\varphi \left( M_{[\mathbf{f}]}(\mathbf{x}) - [\mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x})]_i \right) \\ &\leq \omega_\varphi \left( \frac{1}{\alpha K} \mu^{\frac{2^n}{\exp(K(\max \mathbf{x} - \min \mathbf{x}) - 1)}} \right) \end{aligned}$$

□

**Remark.** Value of  $\varphi$  could be indentify as a value of  $F$  on a diagonal.

**3.2. Arithmetic-Geometric mean.** Arithmetic-geometric means was considered first time by Gauss' in 1870s [5]. In our setting, we define

$$f_1: \mathbb{R}_+ \ni x \mapsto x, \quad f_2: \mathbb{R}_+ \ni x \mapsto \ln(x)$$

and its product  $\mathbf{f} = (f_1, f_2)$ . Then  $\mathbf{A}_{[\mathbf{f}]}(a, b) = (\frac{1}{2}(a + b), \sqrt{ab})$  and there exists a unique function  $M_{[\mathbf{f}]}: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  satisfying  $M_{[\mathbf{f}]} \circ \mathbf{A}_{[\mathbf{f}]} = M_{[\mathbf{f}]}$  and  $\min(a, b) \leq M_{[\mathbf{f}]}(a, b) \leq \max(a, b)$  for any  $a, b \in \mathbb{R}_+^2$ . By uniqueness of  $M_{[\mathbf{f}]}$  it coincides with  $\mathcal{AGM}$ .

Fix  $x_1, x_2 \in \mathbb{R}_+$ ,  $x_1 < x_2$ ,  $\mathbf{x} = (x_1, x_2)$ . We will be interested in estimating  $\max \mathbf{A}_{[\mathbf{f}]}^n(x_1, x_2) - \min \mathbf{A}_{[\mathbf{f}]}^n(x_1, x_2)$ . We have already known inequality (2.2). To visualise our result we will apply Theorem 2 in the spirit of section 2.3.

We have  $P_{f_1}(x) = 0$  and  $P_{f_2}(x) = -1/x$ . Let

$$K := \sup_{\substack{x \in [x_1, x_2] \\ i \in \{1, 2\}}} |P_{f_i}(x)| = \frac{1}{x_1}.$$

Moreover

$$\begin{aligned} n_0 &= \log_2(e) \cdot K \cdot (\max \mathbf{x} - \min \mathbf{x}) - \log_2(e^\xi - 1) + 1 \\ &= \log_2(e) \cdot \frac{1}{x_1} \cdot (x_2 - x_1) - \log_2(e^\xi - 1) + 1 \\ &= \log_2(e) \cdot \frac{x_2}{x_1} - \log_2 e - \log_2(e^\xi - 1) + 1 \\ &\approx 1.44 \frac{x_2}{x_1} + 3.80. \end{aligned}$$

For  $n > n_0$  one has

$$\begin{aligned} \max \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) &< \frac{x_1}{\alpha} \mu^{\frac{2^n}{\exp\left(\frac{1}{x_1}(x_2-x_1)\right)-1}} \\ &= \frac{x_1}{\alpha} \mu^{\frac{2^n}{\exp\left(\frac{x_2}{x_1}-1\right)-1}}. \end{aligned}$$

**Remark.** *Inequality above remains valid (with the same value of  $n_0$ ) if  $\mathbf{A}_{[\mathbf{f}]}$  is a composition of any power means of indexes between 0 and 2 and any number of this means. It particular it holds for a number of classical means: arithmetic-quadratic, quadratic-geometric, arithmetic-geometric-quadratic etc.*

#### 4. AUXILIARY RESULTS

**4.1. Assumption**  $K = 1$ . For fixed  $K > 0$  and interval  $I$  we define an operator  $*$ :  $\mathcal{S}_K(I) \rightarrow \mathcal{S}_1(K \cdot I)$  given by  $*$ :  $f(x) \mapsto f(\frac{x}{K})$ . Then  $P_{f*}(x) = \frac{1}{K}P_f(\frac{x}{K})$ . Moreover

$$A_{[f]}(\mathbf{x}) = \frac{1}{K}A_{[f*]}(K \cdot \mathbf{x}) \text{ for any } \mathbf{x} \in I^n, n \in \mathbb{N}.$$

Whence, for  $\mathbf{f} = (f_1, \dots, f_k)$ ,  $f_i \in \mathcal{S}_K(I)$  and  $\mathbf{f}^* := (f_1^*, \dots, f_k^*)$ ,

$$A_{[\mathbf{f}]}(\mathbf{x}) = \frac{1}{K}A_{[\mathbf{f}^*]}(K \cdot \mathbf{x}) \text{ for any } \mathbf{x} \in I^n,$$

thus, iterating  $A_{[\mathbf{f}]}^n(\mathbf{x}) = \frac{1}{K^n}A_{[\mathbf{f}^*]}^n(K \cdot \mathbf{x})$  for any  $\mathbf{x} \in I^n$ ,  $n \in \mathbb{N}$ .

So

$$\begin{aligned} \max \mathbf{x} - \min \mathbf{x} &= \frac{1}{K}(\max K\mathbf{x} - \min K\mathbf{x}) \\ \max A_{[\mathbf{f}]}^n(\mathbf{x}) - \min A_{[\mathbf{f}]}^n(v) &= \frac{1}{K^n}(\max A_{[\mathbf{f}^*]}^n(K \cdot \mathbf{x}) - \min A_{[\mathbf{f}^*]}^n(K \cdot \mathbf{x})) \end{aligned}$$

Whence in proofs Theorem 1 and Theorem 2 we can assume, with no loss of generality,  $K = 1$ .

**4.2. Single vector results.** Until the end of this section we will be working toward a single vector  $\mathbf{x} \in I^k$  for fix  $k \in \mathbb{N}$ ,  $I$  - an interval. For a continuous, monotone function  $s: I \rightarrow \mathbb{R}$  we adopt some conventions in the spirit of probability theory. Let us denote  $\bar{\mathbf{x}} := A(\mathbf{x})$ . We will also use notions

$$\begin{aligned} A^\mathbf{x}(s(\mathbf{x})) &:= s(A_{[s]}(\mathbf{x})) = \frac{1}{k}(s(\mathbf{x}_1) + s(\mathbf{x}_2) + \dots + s(\mathbf{x}_k)), \\ \text{Var}(\mathbf{x}) &:= A^\mathbf{x}((\mathbf{x} - \bar{\mathbf{x}})^2) = A^\mathbf{x}(\mathbf{x}^2) - \bar{\mathbf{x}}^2. \end{aligned}$$

In this convention  $A^\mathbf{x}(\cdot)$  is a linear operator. Note that functions belonging to  $\mathcal{S}(I)$  are [only] twice differentiable. However in some lemmas below it would be handy to use third derivative. To avoid this drawback, we turn into the convention of Riemann-Stieltjes integral (see Lemma 4.1 below).

**Remark.** *It is just one of possible solutions - otherwise we could consider functions belonging to  $\mathcal{C}^\infty(I) \cap \mathcal{S}(I)$  only, and use some density argument to extend Theorem 1 and Theorem 2 to whole space  $\mathcal{S}(I)$ .*

Now we are going to calculate some integral form of  $A_{[f]}$ . Later, we will majorize most of terms on the right hand side to obtain an approximate value of  $A_{[f]}$  (see Corollary 4.1 below).

**Lemma 4.1.** *Let  $I$  be an interval,  $f \in \mathcal{S}(I)$  and  $\mathbf{x} \in I^k$  for some  $k \in \mathbb{N}$ . Then*

$$A_{[f]}(\mathbf{x}) = \bar{\mathbf{x}} + \frac{1}{2} \text{Var}(\mathbf{x}) P_f(\bar{\mathbf{x}}) + \frac{1}{2f'(\bar{\mathbf{x}})} A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} (\mathbf{x} - t)^2 df''(t) \right) \\ + \int_{\bar{\mathbf{x}}}^{A_{[f]}(\mathbf{x})} \frac{(f(u) - f(A_{[f]}(\mathbf{x}))) f''(u)}{f'(u)^2} du.$$

*Proof.* By Taylor's theorem applied to function  $f$  at  $\bar{\mathbf{x}}$  and function  $f^{-1}$  at  $f(\bar{\mathbf{x}})$  in both cases with integral rest (cf. [4, equation 2.4]) we obtains

$$f(x) = f(\bar{\mathbf{x}}) + (x - \bar{\mathbf{x}}) f'(\bar{\mathbf{x}}) + (x - \bar{\mathbf{x}})^2 \frac{f''(\bar{\mathbf{x}})}{2} + \int_{\bar{\mathbf{x}}}^x \frac{1}{2} (x - t)^2 df''(t), \\ f^{-1}(f(\bar{\mathbf{x}}) + \delta) = \bar{\mathbf{x}} + \frac{\delta}{f'(\bar{\mathbf{x}})} + \int_{f(\bar{\mathbf{x}})}^{f(\bar{\mathbf{x}}) + \delta} \frac{(t - (f(\bar{\mathbf{x}}) + \delta)) f''(f^{-1}(t))}{f'(f^{-1}(t))^3} dt.$$

So

$$f(\mathbf{x}_i) = f(\bar{\mathbf{x}}) + (\mathbf{x}_i - \bar{\mathbf{x}}) f'(\bar{\mathbf{x}}) + (\mathbf{x}_i - \bar{\mathbf{x}})^2 \frac{f''(\bar{\mathbf{x}})}{2} + \int_{\bar{\mathbf{x}}}^{\mathbf{x}_i} \frac{1}{2} (\mathbf{x}_i - t)^2 df''(t),$$

but  $A^{\mathbf{x}}(\mathbf{x} - \bar{\mathbf{x}}) = 0$ , whence

$$A^{\mathbf{x}}(f(\mathbf{x})) = f(\bar{\mathbf{x}}) + A^{\mathbf{x}}((\mathbf{x} - \bar{\mathbf{x}})^2) \frac{f''(\bar{\mathbf{x}})}{2} + A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} \frac{1}{2} (\mathbf{x} - t)^2 df''(t) \right) \\ = f(\bar{\mathbf{x}}) + \text{Var}(\mathbf{x}) \cdot \frac{f''(\bar{\mathbf{x}})}{2} + A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} \frac{1}{2} (\mathbf{x} - t)^2 df''(t) \right).$$

Let us now consider

$$\delta = A^{\mathbf{x}}(f(\mathbf{x})) - f(\bar{\mathbf{x}}) = \text{Var}(\mathbf{x}) \cdot \frac{f''(\bar{\mathbf{x}})}{2} + A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} \frac{1}{2} (\mathbf{x} - t)^2 df''(t) \right),$$

then

$$f^{-1}(A^{\mathbf{x}}(f(\mathbf{x}))) = \bar{\mathbf{x}} + \text{Var}(\mathbf{x}) \cdot \frac{f''(\bar{\mathbf{x}})}{2f'(\bar{\mathbf{x}})} + \frac{1}{f'(\bar{\mathbf{x}})} A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} \frac{1}{2} (\mathbf{x} - t)^2 df''(t) \right) \\ + \int_{f(\bar{\mathbf{x}})}^{A^{\mathbf{x}}(f(\mathbf{x}))} \frac{(t - A^{\mathbf{x}}(f(\mathbf{x}))) f''(f^{-1}(t))}{f'(f^{-1}(t))^3} dt.$$



Upon putting  $t = f(u)$  one has  $dt = f'(u)du$ . Lastly

$$\begin{aligned} A_{[f]}(\mathbf{x}) &= \bar{\mathbf{x}} + \text{Var}(\mathbf{x}) \cdot \frac{f''(\bar{\mathbf{x}})}{2f'(\bar{\mathbf{x}})} + \frac{1}{f'(\bar{\mathbf{x}})} A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} \frac{1}{2}(\mathbf{x} - t)^2 df''(t) \right) \\ &\quad + \int_{\bar{\mathbf{x}}}^{A_{[f]}(\mathbf{x})} \frac{(f(u) - f(A_{[f]}(\mathbf{x})))f''(u)}{f'(u)^2} du. \end{aligned}$$

□

In the next lemma we are going to majorize two right-most terms in Lemma 4.1.

**Lemma 4.2.** *Let  $I$  be an interval,  $f \in \mathcal{S}_K(I)$  for some  $K \in (0, +\infty)$  and  $\mathbf{x} \in I^k$  for some  $k \in \mathbb{N}$ . Then*

$$(i) \left| \int_{\bar{\mathbf{x}}}^{A_{[f]}(\mathbf{x})} \frac{(f(u) - f(A_{[f]}(\mathbf{x})))f''(u)}{f'(u)^2} du \right| < K \cdot (A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}})^2 \exp(\|P_f\|_*),$$

$$(ii) \left| \frac{1}{f'(\bar{\mathbf{x}})} A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} \frac{1}{2}(\mathbf{x} - t)^2 df''(t) \right) \right| \leq \frac{1}{6} \cdot K \cdot \exp(\|P_f\|_*) \cdot A^{\mathbf{x}}(|\mathbf{x} - \bar{\mathbf{x}}|^3),$$

where  $\|P_f\|_* := \sup_{a, b \in I} \left| \int_a^b P_f(t) dt \right|$ .

*Proof.* Let us note that

$$(4.1) \quad \frac{f'(\Omega)}{f'(\Theta)} = \exp\left(\int_{\Theta}^{\Omega} P_f(u) du\right) \leq \exp(\|P_f\|_*), \text{ for any } \Omega, \Theta \in I.$$

(i) We simply calculate

$$\begin{aligned} & \left| \int_{\bar{\mathbf{x}}}^{A_{[f]}(\mathbf{x})} \frac{(f(u) - f(A_{[f]}(\mathbf{x})))f''(u)}{f'(u)^2} du \right| \\ & \leq K \cdot \int_{\bar{\mathbf{x}}}^{A_{[f]}(\mathbf{x})} \left| \frac{(f(u) - f(A_{[f]}(\mathbf{x})))}{f'(u)} \right| du \\ & = K \cdot |A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}}| \left| \frac{(f(\Theta) - f(A_{[f]}(\mathbf{x})))}{f'(\Theta)} \right| \text{ for some } \Theta \in (\bar{\mathbf{x}}, A_{[f]}(\mathbf{x})) \\ & = K \cdot |A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}}| \left| \frac{(\Theta - A_{[f]}(\mathbf{x}))f'(\Omega)}{f'(\Theta)} \right| \text{ for some } \Omega \in (\bar{\mathbf{x}}, A_{[f]}(\mathbf{x})) \\ & \leq K \cdot (A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}})^2 \frac{f'(\Omega)}{f'(\Theta)} \\ & \leq K \cdot (A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}})^2 \exp(\|P_f\|_*) \end{aligned}$$

(ii) By mean value theorem, for any entry  $x$  of  $\mathbf{x}$  there exists  $\beta_x \in (\bar{\mathbf{x}}, x)$  satisfying

$$\int_{\bar{\mathbf{x}}}^x \frac{1}{2}(x - t)^2 df''(t) = \frac{f''(\beta_x)}{2} \int_{\bar{\mathbf{x}}}^x (x - t)^2 dx.$$

Applying mean value theorem again, there exists a universal  $\beta \in (\min \mathbf{x}, \max \mathbf{x})$  satisfying

$$A^{\mathbf{x}} \left( \frac{f''(\beta_{\mathbf{x}})}{2} \int_{\bar{\mathbf{x}}}^{\mathbf{x}} (\mathbf{x} - t)^2 dt \right) = \frac{f''(\beta)}{2} A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} (\mathbf{x} - t)^2 dt \right).$$

Lastly

$$\begin{aligned} \left| \frac{1}{f'(\bar{\mathbf{x}})} A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} \frac{1}{2} (\mathbf{x} - t)^2 df''(t) \right) \right| &= \left| \frac{1}{f'(\bar{\mathbf{x}})} A^{\mathbf{x}} \left( \frac{f''(\beta_{\mathbf{x}})}{2} \int_{\bar{\mathbf{x}}}^{\mathbf{x}} (\mathbf{x} - t)^2 dt \right) \right| \\ &= \left| \frac{f''(\beta)}{2f'(\bar{\mathbf{x}})} A^{\mathbf{x}} \left( \int_{\bar{\mathbf{x}}}^{\mathbf{x}} (\mathbf{x} - t)^2 dt \right) \right| \\ &= \left| \frac{f''(\beta)}{6f'(\bar{\mathbf{x}})} A^{\mathbf{x}} ((\mathbf{x} - \bar{\mathbf{x}})^3) \right| \\ &\leq \frac{1}{6} \left| \frac{f''(\beta)}{f'(\bar{\mathbf{x}})} \right| A^{\mathbf{x}} (|\mathbf{x} - \bar{\mathbf{x}}|^3) \\ &= \frac{1}{6} \left| \frac{f''(\beta)}{f'(\beta)} \right| \left| \frac{f'(\beta)}{f'(\bar{\mathbf{x}})} \right| A^{\mathbf{x}} (|\mathbf{x} - \bar{\mathbf{x}}|^3) \\ &\leq \frac{1}{6} \cdot K \cdot \exp(\|P_f\|_*) \cdot A^{\mathbf{x}} (|\mathbf{x} - \bar{\mathbf{x}}|^3) \end{aligned}$$

□

Now, by applying Lemma 4.2 to Lemma 4.1, we obtain the following

**Corollary 4.1.** *Let  $I$  be an interval,  $f \in \mathcal{S}(I)$  and  $\mathbf{x} \in I^k$  for some  $k \in \mathbb{N}$ ,  $\|P_f\|_{\infty} < K$ . Then*

$$|A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}} - \frac{1}{2} \text{Var}(\mathbf{x}) P_f(\bar{\mathbf{x}})| < K \cdot \exp(\|P_f\|_*) \cdot ((A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}})^2 + \frac{1}{6} A^{\mathbf{x}} (|\mathbf{x} - \bar{\mathbf{x}}|^3))$$

Therefore, the value of quasi-arithmetic mean could be approximated  $A_{[f]}(\mathbf{x}) \approx \bar{\mathbf{x}} + \frac{1}{2} \text{Var}(\mathbf{x}) P_f(\bar{\mathbf{x}})$ . Such an informal expression could be predicted much earlier - after Proposition 2.1. The only parameters to calculate was  $\frac{1}{2} \text{Var}(\mathbf{x})$  multiplying  $P_f(\bar{\mathbf{x}})$  and the majorization of error, which was the most difficult part.

In this moment we reiterate that our aim is to describe whole family [in particular all results concerning means its built up] by a single parameter -  $K$ .

If the difference between maximal and minimal entry of vector  $\mathbf{x}$  is small enough we would like to approximate  $A_{[f]}(\mathbf{x}) \approx \bar{\mathbf{x}}$  (Lemma 4.3). If this difference is too big, we will use property (2.3) to decrease it (Lemma 4.4) - it is applicable for any vector but gives worse estimation. The main idea of the proof of Theorem 1 is to apply Lemma 4.4 by a number of steps to fulfilled the assumption of Lemma 4.3 and later apply this lemma.

**Lemma 4.3.** *Let  $I$  be an interval,  $f \in \mathcal{S}_K(I)$  and  $\mathbf{x} \in I^k$  for some  $k \in \mathbb{N}$  and  $K \in (0, +\infty)$ .*

*If  $\max \mathbf{x} - \min \mathbf{x} < \min(1/K, 1)$  then*

$$|A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}}| < \frac{\alpha}{2} \cdot K \cdot (\max \mathbf{x} - \min \mathbf{x})^2,$$

*where  $\alpha = \frac{3+7e}{3}$ .*

*Proof.* Let  $\delta := \max \mathbf{x} - \min \mathbf{x}$ . By the definition

$$\text{Var}(\mathbf{x}) = A^{\mathbf{x}}((\mathbf{x} - \bar{\mathbf{x}})^2) < \delta^2.$$

Similarly  $(A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}})^2 < \delta^2$  and  $A^{\mathbf{x}}(|\mathbf{x} - \bar{\mathbf{x}}|^3) < \delta^3$ .

We will restrict interval  $I$  to  $J := [\min \mathbf{x}, \max \mathbf{x}] \subset I$ . Then we consider  $h = f|_J \in \mathcal{S}_K(J)$ ,  $\|P_h\| \leq \delta K$ . By Corollary 4.1 applied to  $h$ , we obtain

$$\begin{aligned} |A_{[f]}(\mathbf{x}) - \bar{\mathbf{x}}| &= |A_{[h]}(\mathbf{x}) - \bar{\mathbf{x}}| \\ &< \frac{1}{2} \text{Var}(\mathbf{x}) |P_h(\bar{\mathbf{x}})| + K \cdot \exp(\|P_h\|_*) \cdot ((A_{[h]}(\mathbf{x}) - \bar{\mathbf{x}})^2 + \frac{1}{6} A^{\mathbf{x}}(|\mathbf{x} - \bar{\mathbf{x}}|^3)) \\ &< \frac{1}{2} \delta^2 K + K \cdot e^{\delta K} (\delta^2 + \frac{1}{6} \delta^3) \\ &\leq \frac{1}{2} \delta^2 K + K \cdot \frac{7e}{6} \delta^2 \\ &\leq \frac{3+7e}{6} \delta^2 K \end{aligned}$$

□

**Lemma 4.4.** *Let  $k \in \mathbb{N}$ ,  $\mathbf{x} \in \mathbb{R}^k$  and  $K > 0$ . Then*

$$\exp(K \cdot (\mathcal{E}_K(\mathbf{x}) - \mathcal{E}_{-K}(\mathbf{x}))) - 1 \leq \frac{1}{2} \left( \exp(K \cdot (\max \mathbf{x} - \min \mathbf{x})) - 1 \right).$$

*Proof.* Let us assume  $\mathbf{x}_1 \leq \mathbf{x}_2 \leq \dots \leq \mathbf{x}_k$ . Then, by simple transformations,

$$\begin{aligned} \mathcal{E}_K(\mathbf{x}) - \mathcal{E}_{-K}(\mathbf{x}) &= \frac{1}{K} \ln \left( \frac{\sum \exp(K \cdot \mathbf{x}_i)}{k} \frac{\sum \exp(-K \cdot \mathbf{x}_i)}{k} \right); \\ e^{K(\mathcal{E}_K(\mathbf{x}) - \mathcal{E}_{-K}(\mathbf{x}))} &= \frac{1}{k^2} \sum_{i=1}^k e^{K \cdot \mathbf{x}_i} \sum_{j=1}^k e^{-K \cdot \mathbf{x}_j} = \frac{1}{k^2} \sum_{i=1}^k \sum_{j=1}^k e^{K \cdot (\mathbf{x}_i - \mathbf{x}_j)}; \\ e^{K(\mathcal{E}_K(\mathbf{x}) - \mathcal{E}_{-K}(\mathbf{x}))} - 1 &= \frac{1}{k^2} \sum_{i=1}^k \sum_{j=1}^k (e^{K \cdot (\mathbf{x}_i - \mathbf{x}_j)} - 1). \end{aligned}$$

Now we may omit  $(e^{K \cdot (\mathbf{x}_i - \mathbf{x}_j)} - 1)$  for  $i \leq j$  - these elements are non-positive so the sum does not decrease. Later we will majorize  $\mathbf{x}_i - \mathbf{x}_j \leq \max \mathbf{x} - \min \mathbf{x}$ .

$$\begin{aligned} e^{K(\mathcal{E}_K(\mathbf{x}) - \mathcal{E}_{-K}(\mathbf{x}))} - 1 &\leq \frac{1}{k^2} \sum_{i>j} (e^{K \cdot (\mathbf{x}_i - \mathbf{x}_j)} - 1) \\ &\leq \frac{1}{k^2} \frac{k(k-1)}{2} (e^{K \cdot (\max \mathbf{x} - \min \mathbf{x})} - 1) \\ &\leq \frac{1}{2} (e^{K \cdot (\max \mathbf{x} - \min \mathbf{x})} - 1) \end{aligned}$$

□

## 5. PROOFS OF THEOREM 1 AND THEOREM 2

**5.1. Proof of Theorem 1.** Let  $\mathbf{x} \in I^k$  and, by section 4.1,  $f_i \in \mathcal{S}_1(I)$  for any  $i \in \{1, \dots, k\}$ . By (2.3) and Lemma 4.4 we have

$$\begin{aligned} \exp\left(K\left(\max \mathbf{A}_{[\mathbf{f}]}(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}(\mathbf{x})\right)\right) - 1 &\leq \exp(\mathcal{E}_1(\mathbf{x}) - \mathcal{E}_{-1}(\mathbf{x})) - 1 \\ &\leq \frac{1}{2} \left(e^{\max \mathbf{x} - \min \mathbf{x}} - 1\right). \end{aligned}$$

So, by simple induction, using definition of  $n_0$ , one has

$$\exp\left(\max \mathbf{A}_{[\mathbf{f}]}^{n_0}(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^{n_0}(\mathbf{x})\right) - 1 \leq \frac{1}{2^{n_0}} \left(e^{\max \mathbf{x} - \min \mathbf{x}} - 1\right) \leq e^l - 1.$$

Whence

$$(5.1) \quad \max \mathbf{A}_{[\mathbf{f}]}^{n_0}(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^{n_0}(\mathbf{x}) < l.$$

Therefore the conjecture is satisfied for  $n = n_0$ . Moreover  $l < 1$  is making Lemma 4.3 applicable since  $n_0$ -th iteration. For  $n \geq n_0$ , we obtain

$$\begin{aligned} \max \mathbf{A}_{[\mathbf{f}]}^{n+1}(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^{n+1}(\mathbf{x}) &\leq \left| \max \mathbf{A}_{[\mathbf{f}]}^{n+1}(\mathbf{x}) - \bar{\mathbf{x}} \right| + \left| \bar{\mathbf{x}} - \min \mathbf{A}_{[\mathbf{f}]}^{n+1}(\mathbf{x}) \right| \\ (5.2) \quad &\leq \alpha \cdot (\max \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}))^2. \end{aligned}$$

This inequality is related with (2.2) for arithmetic-geometric means. By simple induction, using inequalities (5.1) and (5.2), we obtain

$$\max \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) - \min \mathbf{A}_{[\mathbf{f}]}^n(\mathbf{x}) < \frac{1}{\alpha} (\alpha l)^{2^{n-n_0}} \text{ for any } n \geq n_0.$$

**5.2. Proof of Theorem 2.** Putting  $l = \xi$  in Theorem 1 and recalling an assumption  $K = 1$ , we get

$$\begin{aligned} n_0 &= \left\lceil \log_2 \left( \frac{\exp(\max \mathbf{x} - \min \mathbf{x}) - 1}{e^\xi - 1} \right) \right\rceil \\ &= \left\lceil \log_2(\exp(\max \mathbf{x} - \min \mathbf{x}) - 1) - \log_2(e^\xi - 1) \right\rceil \\ &\leq \left\lceil \log_2(\exp(\max \mathbf{x} - \min \mathbf{x})) - \log_2(e^\xi - 1) \right\rceil \\ &= \left\lceil \log_2(e) \cdot (\max \mathbf{x} - \min \mathbf{x}) - \log_2(e^\xi - 1) \right\rceil \\ &< \log_2(e) \cdot (\max \mathbf{x} - \min \mathbf{x}) - \log_2(e^\xi - 1) + 1 =: n_1. \end{aligned}$$

Approximately  $n_1 \approx 1.4427 \cdot (\max \mathbf{x} - \min \mathbf{x}) + 5.246$ .

Then, for  $n \geq n_1$  (simultaneously  $n \geq n_0$ ),

$$\begin{aligned}
 \max \mathbf{A}_{[f]}^n(\mathbf{x}) - \min \mathbf{A}_{[f]}^n(\mathbf{x}) &< \frac{1}{\alpha} (\alpha l)^{2^{n-n_0}} \\
 &< \frac{1}{\alpha} \left( (\alpha \xi)^{e^\xi - 1} \right)^{\frac{2^{n-1}}{\exp(\max \mathbf{x} - \min \mathbf{x}) - 1}} \\
 &= \frac{1}{\alpha} \left( (\alpha \xi)^{(e^\xi - 1)/2} \right)^{\frac{2^n}{\exp(\max \mathbf{x} - \min \mathbf{x}) - 1}} \\
 &= \frac{1}{\alpha} \mu^{\frac{2^n}{\exp(\max \mathbf{x} - \min \mathbf{x}) - 1}}.
 \end{aligned}$$

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